Exact WKB analysis and Jacobi polynomials (New development of microlocal analysis and singular perturbation theory)

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By

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Abstract

An asymptotic formula for the Jacobi polynomials given by Kuijlaars and Martínez-Finkelshtein [17] is reproduced and some generalizations are given from the viewpoint of exact WKB analysis. Asymptotic series of all orders are obtained and their assumptions for parameters are understood by the classification of Stokes geometry of the hypergeometric differential equation.

§ 1. Introduction

The aim of this article is to give a new proof and some generalizations of an asymptotic formula of the Jacobi polynomials given by Kuijlaars and Martínez-Finkelshtein [17] from the viewpoint of exact WKB analysis. Original proof given in [17] of the formula is based on the non-linear steepest descent analysis of Deift-Zhou [9]. In our forthcoming paper [1], the relations between the hypergeometric function and WKB solutions are established. We prove their formula as an application of some formulas to be given in [1]. Some parts of the results of [1] have been announced in [2]. See [3], [5] also.

In [17], asymptotic behavior of monic Jacobi polynomials $\hat{P}_n^{(\alpha,\beta)}(x)$ as $n \to \infty$ was investigated with the varying parameters

$$\alpha = nA, \quad \beta = nB$$

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under the conditions $-1 < A < 0,\ -1 < B < 0,\ -2 < A+B < -1$. On the other hand, the hypergeometric function $F(\alpha_0+\alpha\eta, \beta_0+\beta\eta, \gamma_0+\gamma\eta; x)$ with a large parameter η can be written down in terms of the Borel sums of WKB solutions to the hypergeometric differential equation with the large parameter ([1], [2], [3]). The Jacobi polynomials are obtained by taking special parameters in the hypergeometric function. Note that our large parameter η stands for the parameter n in [17]. Thus we can write down the polynomial as a linear combination of the Borel resummed WKB solutions. Using Watson's lemma, we have an asymptotic expansion formula of the Jacobi polynomials by WKB solutions. We show that the leading terms of this obtained asymptotic series coincide with the asymptotic formula given in [17].

There are many works on Jacobi polynomials (for example, see [10], [13], [18], [19], [20], [22] and the references cited in these articles). Roughly speaking, main interests of these articles are zeros of the polynomials and their asymptotic behvior and various methods are used. In [17], Martínez-Finkelshtein et al. study the asymptotic behavior of the polynomials and as an application, they find the accumulation point set of zeros as $n \to \infty$. As is mentioned above, the Jacobi polynomials are the special case of the hypergeometric function and our recent works yield asymptotic behavior of the hypergeometric function with respect to the parametes systematially. There are a number of works on asymptotic behavior of the hypergeometric function with respect to the parametes [12], [24], however, these articles consider the case where the Stokes geometry is degenerate in the sense that there exists a Stokes curve which flows into a turning point [4], [15].

There are several advantages of our analysis. Firstly, as is written above, we have obtained asymptotic formulas not only for Jacobi polynomials but for the hypergeometric function with respect to the large parameter. These asymptotic formulas as $n \to \infty$ or $\eta \to \infty$ contain asymptotics for all orders. By computing WKB solutions, we have asymptotic series of the Jacobi polynomials (or the hypergeometric function) with respect to the powers of n^{-1} (or of η^{-1}) as many as one likes both for dominant asymptotics and for exponentially small terms. Our method can be applied to any orthogonal polynomial coming from the hypergeometric function and the confluent hypergeometric functions. Secondly, the asymptotic formulas as $n \to \infty$ or $\eta \to \infty$ in each cases described in Figure 1 in [17] can be obtained in a unified manner by using the exact WKB analysis. Thirdly, a variety of asymptotic formulas given in [17] and in references cited in [17] is understood by the types of Stokes geometry of the hypergeometric differential equation which are classified in terms of the parameters of hypergeometric differential equation (cf. [4]). The relation between pairs of asymptotic formulas corresponding to adjacent regions of parameters discussed in [4] can be explained by the parametric Stokes phenomena investigated in [5].

§ 2. Jacobi polynomials

The classical Jacobi polynomial is defined by the following equation [22]:

$$\begin{aligned}P_n^{(\alpha,\beta)}(x) &= \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\{(1-x)^{\alpha+n}(1+x)^{\beta+n}\}\\ &= l_n^{(\alpha,\beta)}x^n + \cdots \ (x \in \mathbb{C}),\end{aligned}$$

where $l_n^{(\alpha,\beta)}$ denotes the leading coefficient defined as follows:

$$l_n^{(\alpha,\beta)} = 2^{-n}\frac{\Gamma(2n+\alpha+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}.$$

The monic Jacobi polynomial is obtained by dividing $P_n^{(\alpha,\beta)}(x)$ by $l_n^{(\alpha,\beta)}$. We let $\hat{P}_n^{(\alpha,\beta)}(x)$ denote the monic Jacobi polynomial:

$$\hat{P}_n^{(\alpha,\beta)}(x) = 2^n\frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)}P_n^{(\alpha,\beta)}(x). \tag{2.1}$$

It is well known that the classical Jacobi polynomial is represented by the Gauss hypergeometric function:

$$\begin{aligned}P_n^{(\alpha,\beta)}(x) &= \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}F(-n, n+\alpha+\beta+1, \alpha+1; \frac{1-x}{2})\\ &= 2^{-n}\frac{\Gamma(2n+\alpha+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}x^n + \cdots .\end{aligned}$$

Here $F(a,b,c;z)$ is the Gauss hypergeometric function defined by the hypergeometric series:

$$F(a,b,c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}.$$

Here we set $(s)_n = \Gamma(s+n)/\Gamma(s)$ for a complex number. For the special choices of parameters $a=-n,\ b=n+\alpha+\beta+1,\ c=\alpha+1,\ F(a,b,c;z)$ satisfies the following Gauss hypergeometric differential equation

$$z(1-z)\frac{d^2y}{dz^2} + (\alpha+1-(\alpha+\beta+2)z)\frac{dy}{dz} + n(n+\alpha+\beta+1)y = 0. \tag{2.2}$$

Similarly, $\hat{P}_n^{(\alpha,\beta)}(x)$ can be represented as

$$\hat{P}_n^{(\alpha,\beta)}(x) = 2^n\frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)\Gamma(\alpha+1)}F(-n, n+\alpha+\beta+1, \alpha+1; \frac{1-x}{2}).$$

Asymptotic formulas for $\hat{P}_n^{(\alpha,\beta)}(x)$ are given in [17] when $n\to\infty$ with $\alpha = nA,\ \beta = nB\ (-1<A<0, -1<B<0, -2<A+B<-1)$, which are called *nonstandard* *parameters.* The representation of $\hat{P}_n^{(nA,nB)}(x)$ in terms of the hypergeometric function is as follows:

$$\begin{aligned}\hat{P}_n^{(nA,nB)}(x) = 2^n \frac{\Gamma(n(A+1)+1)\Gamma(n(A+B+1)+1)}{\Gamma(n(A+B+2)+1)\Gamma(nA+1)} \\ \times F(-n, n(A+B+1)+1, nA+1; \frac{1-x}{2}).\end{aligned} \tag{2.3}$$

§ 3. Exact WKB analysis for the hypergeometric differential equation

In this section, we define WKB solutions for the following hypergeometric differential equation:

$$z(1-z)\frac{d^2y}{dz^2} + (\eta A + 1 - (\eta(A+B)+2)z)\frac{dy}{dz} + \eta(\eta(A+B+1)+1)y = 0, \tag{3.1}$$

where η is a large parameter. The equation (3.1) is obtained by replacing α and β by ηA and ηB respectively in (2.2). To analyze (3.1) by using the standard theory of exact WKB analysis [15], we eliminate the first order term of (3.1) by the following transformation:

$$y = z^{-\frac{1}{2}(1+\eta A)}(1-z)^{-\frac{1}{2}(1+\eta B)}\psi.$$

and we get

$$\Big(-\frac{d^2}{dz^2} + \eta^2 Q(z,\eta)\Big)\psi = 0, \tag{3.2}$$

where $Q(z,\eta) = Q_0(z) + \eta^{-1}Q_1(z) + \eta^{-2}Q_2(z)$ with

$$\begin{aligned} Q_0(z) =& \frac{(A+B+2)^2z^2 + 2(2(A+B+1)+A^2+AB)z + A^2}{4z^2(z-1)^2}, \\ Q_1(z) =& \frac{A+B+2}{2z(z-1)}, \\ Q_2(z) =& -\frac{1}{4z^2(z-1)^2}. \end{aligned} \tag{3.3}$$

the function $Q_0(z)$ has two zeros z_i $(i = 0, 1)$:

$$z_0 = \frac{(A+B+2)^2 + A^2 - B^2 - 4i\sqrt{(A+1)(B+1)(-A-B-1)}}{2(A+B+2)^2},$$

$$z_1 = \frac{(A+B+2)^2 + A^2 - B^2 + 4i\sqrt{(A+1)(B+1)(-A-B-1)}}{2(A+B+2)^2}$$

Under the assumption in the next section (i.e. $(A,B) \in D$), $z_0 \neq z_1$ and they are simple zeros. The zeros z_0 and z_1 are usually called the simple turning points in the exact WKB analysis. The equation (3.2) has the following formal solutions:

$$\begin{aligned}\psi_\pm(z,\eta) &= \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_{z_0}^{z} S_{\mathrm{odd}} dz\right) \\ &= \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \frac{1}{2} \int_{C_0} S_{\mathrm{odd}} dz\right),\end{aligned}$$

where C_0 is a path of integration starting from z, going around z_0 in a counterclockwise manner and going back to the point of departure (cf. [15]). Also, $S_{\mathrm{odd}} = (S^{(+)} - S^{(-)})/2$, where $S^{(\pm)}$ are the formal solutions $S^{(\pm)} = \sum_{j \geq -1} S_j^{(\pm)} \eta^{-j}$ to the Riccati equation

$$\frac{dS}{dz} + S^2 = \eta^2 Q(z,\eta) \tag{3.4}$$

obtained from (3.2) by setting $S = \psi'/\psi$ and the sign "$\pm$" comes from the choice of the branch of $S_{-1} = \pm\sqrt{Q_0(z)}$. Note that z_i $(i = 0,1)$ coincide with $\zeta_\pm$ in [17] respectively. We call $\psi_\pm$ the WKB solutions normalized at the turning points. In general, the WKB solutions $\psi_\pm$ are divergent series in η^{-1}. To obtain analytic solution of the equation (3.2), we use the Borel resummation method. The constructed true solution by using the Borel resummation method is called the Borel sums of $\psi_\pm$, which are designated by $\Psi_\pm$ below. It is known that under suitable conditions, the WKB solutions $\psi_\pm$ are Borel summable [16], [23]. We give a criterion of the Borel summability which are called Stokes curves emanating from the turning points.

Definition 3.1. Stokes curves emanating from the turning points z_i $(i = 0,1)$ are defined by

$$\mathrm{Im} \int_{z_i}^{z} \sqrt{Q_0} dz = 0.$$

§ 4. Jacobi polynomials and WKB solutions

We assume that (A,B) belongs to

$$D = \{(A,B) \in \mathbb{C}^2 | -1 < \mathrm{Re}\ A < 0,\ -1 < \mathrm{Re}\ B < 0,\ -2 < \mathrm{Re}\ A + \mathrm{Re}\ B < -1\}.$$

Then the Stokes graph of the hypergeometric differential equation has the order sequence $(2,2,2)$ (cf. [4]). This means that there are two Stokes curves flowing into regular singular points 0, 1 and ∞, resppectively. An example of configuration of the Stokes curves of (3.2) under our condition $(A,B) \in D$ is given in Figure 1.

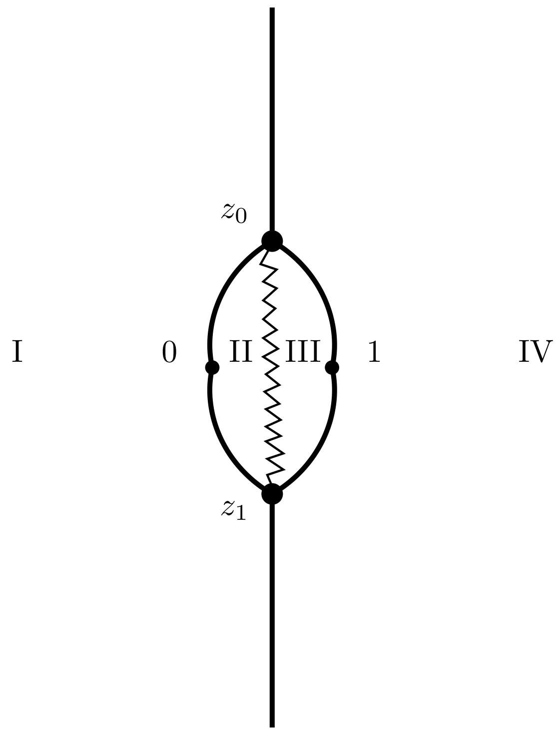

Figure 1. The Stokes curves of (3.2) for $(A, B) = (-0.7, -0.7)$

Here the wiggly line denotes the branch cut for $\sqrt{Q_0(z)}$. Regions whose boundary consists of Stokes curves or the branch cut are labeled respectively by I, II, III and IV shown in Figure 1. Note that if the parameters satisfy $(A, B) \in D$, then z_i $(i = 0, 1)$ are simple zeros of $Q_0(z)$ and the WKB solutions are Borel summable in each region which by Stokes curves (i.e. a Stokes region).

Now we are ready to state our main results.

Theorem 4.1. *Assume that the parameters $(A, B) \in D$, and let $\Psi_\pm^{\mathrm{II}}$ be the Borel sums of $\psi_\pm$ in the region* II. *Then the Gauss hypergeometric function $F(-\eta, \eta(A+B+1)+1, \eta A+1; z)$ is represented by $\Psi_\pm^{\mathrm{II}}$ as follows*:

(4.1)

$$
\begin{aligned}
&F(-\eta, \eta(A+B+1)+1, \eta A+1; z)\\
&= \frac{\Gamma(1-A\eta)\Gamma(-B\eta)\Gamma(1+(1+A)\eta)\Gamma(-(1+B)\eta)}{\Gamma(1+\eta)\Gamma(-(A+B+1)\eta)\Gamma(1+A\eta)\Gamma(-B\eta)}\\
&\times \Bigg[\sqrt{\frac{-A\eta}{2}}\mathrm{e}^{\lambda_1}\Big(\frac{\Gamma(1+\eta(1+A))\Gamma(1-A\eta)\Gamma(-A\eta)}{\Gamma(\eta+1)\Gamma(-(1+B+A)\eta)\Gamma(1+(B+1)\eta)}\Big)^{\frac{1}{2}}\Psi_+^{\mathrm{II}}\\
&+\frac{\mathrm{e}^{\lambda_2}}{i}\sqrt{\frac{-B\eta}{2}}\frac{\Gamma(1-A\eta)\Gamma(B\eta)}{\Gamma(-(1+A+B)\eta)}\Big(\frac{\Gamma(1+\eta(1+B))\Gamma(1-B\eta)\Gamma(-B\eta)}{\Gamma(-\eta(1+A+B)\eta)\Gamma(-(1+B+A)\eta)\Gamma(1+(A+1)\eta)}\Big)^{\frac{1}{2}}\\
&\times\frac{\sin((1+A)\pi\eta)\Gamma(1+(1+A+B)\eta)-\sin(\pi\eta)\Gamma(-(1+B)\eta)}{\sin((1+A)\pi\eta)\Gamma(-(1+A)\eta)\Gamma(1+(1+A+B)\eta)}\Psi_-^{\mathrm{II}}\Bigg],
\end{aligned}
$$

where λ_1 *and* λ_2 *are defined as*

$$\begin{aligned}\lambda_1 =&(-\eta(\frac{1}{2}+A)+\frac{1}{4})\pi i,\\ \lambda_2 =&-(-\eta(\frac{1}{2}+B)+\frac{1}{4})\pi i.\end{aligned}\tag{4.2}$$

As a corollary, we obtain the asymptotic formula for $\hat{P}_n^{(\alpha,\beta)}(1-2z)$ with the aid of the relation (2.3).

Corollary 4.2. *Assume that the parameters* $(A,B)\in D$ *and* z *belongs to the region* II. *the monic Jacobi polynomial* $\hat{P}_n^{(nA,nB)}(1-2z)$ *has the following asymptotic behavior as* $n\to\infty$:

$$\begin{aligned}&\hat{P}_n^{(nA,nB)}(1-2z)\sim\\ &2^n z^{-\frac{1}{2}(1+nA)}(1-z)^{-\frac{1}{2}(1+nB)}\frac{\Gamma(n(1+A+B)+1)\Gamma(-(1+A+B)n)\Gamma(n+1)}{\Gamma(1+n(2+A+B))\Gamma(-(1+B)n)\Gamma(1-An)}\\ &\times\Big[\sqrt{\frac{-An}{2}}e^{\lambda_1}\Big(\frac{\Gamma(1+n(1+A))\Gamma(1-An)\Gamma(-An)}{\Gamma(n+1)\Gamma(-(1+B+A)n)\Gamma(1+(B+1)n)}\Big)^{\frac{1}{2}}\psi_+(z,n)\\ &+\frac{1}{i}\sqrt{\frac{-Bn}{2}}e^{\lambda_2}\Big(\frac{\Gamma(1+n(1+B))\Gamma(1-Bn)\Gamma(-Bn)}{\Gamma(-n(1+A+B)n)\Gamma(-(1+B+A)n)\Gamma(1+(A+1)n)}\Big)^{\frac{1}{2}}\\ &\times\frac{\Gamma(1-An)\Gamma(Bn)}{\Gamma(-(1+A)n)\Gamma(1+(1+B)n)}\psi_-(z,n)\Big]\end{aligned}\tag{4.3}$$

Morover, it turns out that the leading terms of $\psi_\pm(z,n)$ *are*

$$\begin{aligned}\psi_+(z,n)=&\,e^{-\frac{\pi i}{4}}n^{-\frac{1}{2}}2^{-\frac{1}{2}}z^{\frac{1}{2}}(1-z)^{\frac{1}{2}}\frac{(A+B+2)^{\frac{1}{2}}}{((A+1)(B+1)(-A-B-1))^{\frac{1}{4}}}\\ &\times\frac{1}{i}\left(\left(\frac{z-z_1}{z-z_0}\right)^{\frac{1}{4}}-\left(\frac{z-z_0}{z-z_1}\right)^{\frac{1}{4}}\right)\exp\left(n\int_{z_0}^{z}S_{-1}dz\right)\end{aligned}\tag{4.4}$$

$$\begin{aligned}\psi_-(z,n)=&\,e^{-\frac{\pi i}{4}}n^{-\frac{1}{2}}2^{-\frac{1}{2}}z^{\frac{1}{2}}(1-z)^{\frac{1}{2}}\frac{(A+B+2)^{\frac{1}{2}}}{((A+1)(B+1)(-A-B-1))^{\frac{1}{4}}}\\ &\times\left(\left(\frac{z-z_0}{z-z_1}\right)^{\frac{1}{4}}+\left(\frac{z-z_1}{z-z_0}\right)^{\frac{1}{4}}\right)\exp\left(-n\int_{z_0}^{z}S_{-1}dz\right)\end{aligned}\tag{4.5}$$

and (4.3) *yields* (*c*) *of Theorem 2.6 in* [17].

Outline of Proof of Theorem 3.1.

We give an outline of the proof. For details, we refer the readers to [1] where general cases of parameters will be discussed. As is shown in Figure 1, we take a segment connecting two turning points z_0 and z_1 as a branch cut (Figure 1), and take the branch of $\sqrt{Q_0(z)}$ near $z=\infty$ on the first sheet so that

$$\sqrt{Q_0(z)} \sim \frac{A+B+2}{2z}$$

holds. Then the behavior of $\sqrt{Q_0(z)}$ near $z=0$ and $z=1$ are

$$\sqrt{Q_0(z)} \sim -\frac{A}{2z},$$
$$\sqrt{Q_0(z)} \sim -\frac{B}{2(z-1)},$$

respectively. We use the following convention throughout this paper:

$$-A = \mathrm{e}^{-\pi i}A,$$

$$-B = \mathrm{e}^{\pi i}B,$$

$$-(A+1) = \mathrm{e}^{-\pi i}(A+1),$$

$$-(B+1) = \mathrm{e}^{\pi i}(B+1),$$

$$-(A+B+1) = \mathrm{e}^{\pi i}(A+B+1),$$

$$-(A+B+2) = \mathrm{e}^{\pi i}(A+B+2),$$

$$-z = \mathrm{e}^{-\pi i}z$$

and

$$z-1 = \mathrm{e}^{\pi i}(1-z).$$

We define the Voros coefficient and WKB solutions normalized at a singular point.

Definition 4.3. Let b denote one of 0, 1 or ∞. We set

$$V_b = \int_b^{z_i} (S_{\mathrm{odd}} - S_{\mathrm{odd},\le 0})dz,$$

where $S_{\mathrm{odd},\le 0}$ denotes $\eta S_{\mathrm{odd},-1} + S_{\mathrm{odd},0}$. The formal power series V_b is said to be the Voros coefficient at the singular point $z=b$.

We can obtain the explicit form of the Voros coefficient.

Proposition 4.4. *The Voros coefficients V_b $(b=0,1)$ have the following forms*:

$$V_0 = -\frac{1}{2}\sum_{m=1}^{\infty}\frac{\eta^{1-2m}}{2m(2m-1)}B_{2m}\Big(-1+\frac{1}{(1+A+B)^{2m-1}}+\frac{1}{(A+1)^{2m-1}} - \frac{1}{(B+1)^{2m-1}}-\frac{2}{A^{2m-1}}\Big), \tag{4.6}$$

$$V_1 = \frac{1}{2}\sum_{m=1}^{\infty} \frac{\eta^{1-2m}}{2m(2m-1)} B_{2m}\Big(-1 + \frac{1}{(1+A+B)^{2m-1}} - \frac{1}{(A+1)^{2m-1}} + \frac{1}{(B+1)^{2m-1}} + \frac{2}{B^{2m-1}}\Big). \tag{4.7}$$

Here B_n is the Bernoulli numbers defined by

$$\frac{t\mathrm{e}^t}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n$$

(see (1.7) in Section 1.3 in [6]). Note that we can obtain V_b $(b = 0, 1)$ as a special case of the general case which will be discussed in [1] (see also [2] and [5, Theorem 2.3]). Further, we find that the Voros coefficient is Borel summable if $(A, B) \in D$ and we can obtain the explicit form of their Borel sums (see [5, Theorem 3.1] for similar results).

Proposition 4.5. *Let V_b^D $(b = 0, 1)$ denote the Borel sums of Voros coefficients V_b $(b = 0, 1)$. If $(A, B) \in D$, V_b^D $(b = 0, 1)$ have the following forms*:

$$V_0^D = \frac{1}{2}\log \frac{\Gamma(1+(A+1)\eta)\Gamma(-A\eta)\Gamma(1-A\eta)}{\Gamma(\eta+1)\Gamma(-(1+A+B)\eta)\Gamma(1+(B+1)\eta)} \times \frac{(-(1+A+B))^{-\frac{1}{2}-(A+B+1)\eta}(B+1)^{\frac{1}{2}+(B+1)\eta}}{(-A)^{-2A\eta}(A+1)^{\frac{1}{2}+(A+1)\eta}} \tag{4.8}$$

$$V_1^D = \frac{1}{2}\log \frac{\Gamma(1+(B+1)\eta)\Gamma(-B\eta)\Gamma(1-B\eta)}{\Gamma(\eta+1)\Gamma(-(1+A+B)\eta)\Gamma(1+(A+1)\eta)} \times \frac{(B+1)^{-\frac{1}{2}-(B+1)\eta}(A+1)^{\frac{1}{2}+(A+1)\eta}}{(-B)^{-2B\eta}(-(1+A+B))^{\frac{1}{2}+(1+A+B)\eta}} \tag{4.9}$$

Before stating another proposition, we let

$$w_0 = F(-\eta, 1+(A+B+1)\eta, 1+A\eta; z),$$
$$w_1 = z^{-A\eta}F(-(A+1)\eta, 1+(B+1)\eta, 1-A\eta; z).$$

and let

$$u_0 = F(-\eta, 1+(A+B+1)\eta, 1+B\eta; 1-z),$$
$$u_1 = (1-z)^{-B\eta}F(1+(A+1)\eta, -(B+1)\eta, 1-B\eta; 1-z).$$

These pairs of functions (w_0, w_1) and (u_0, u_1) are bases of the solution space of (3.1) near $z = 0$ and $z = 1$, respectively.

Proposition 4.6. *Let $\Psi_{\pm}^{\mathrm{II}}$ be the Borel sums of the WKB solutions in region* II, *then the relation between w_1 and Ψ_{+}^{II} is given by*

$$w_1 = \sqrt{-\frac{A\eta}{2}}\,\mathrm{e}^{\frac{\pi i}{2}(1-A\eta)}\mathrm{e}^{-h_0}\mathrm{e}^{V_0^D}\Psi_{+}^{\mathrm{II}}, \tag{4.10}$$

where

$$\begin{aligned}h_0 = &-\frac{1}{2}(\frac{1}{2}+(1+A+B)\eta)\log(-(1+A+B))\\ &-\frac{1}{2}(\frac{1}{2}+(A+1)\eta)\log(-(A+1))+\frac{1}{2}(\frac{1}{2}+(1+B)\eta)\log(1+B)+A\eta\log(-A).\end{aligned}$$

Similarly, let $\Psi_{\pm}^{\mathrm{IV}}$ *be the Borel sums of the WKB normalized at* $z=z_0$ *in region* IV, *then the relation between* u_1 *and* Ψ_{+}^{IV} *is given by*

$$u_1=\sqrt{-\frac{B\eta}{2}}\mathrm{e}^{-\frac{\pi i}{2}(1-B\eta)}\mathrm{e}^{-h_1}\mathrm{e}^{V_1^D}\Psi_{+}^{\mathrm{IV}}, \tag{4.11}$$

where

$$\begin{aligned}h_1 = &-\frac{1}{2}(\frac{1}{2}+(1+A+B)\eta)\log(-(1+A+B))\\ &+\frac{1}{2}(\frac{1}{2}+(A+1)\eta)\log(-(A+1))-\frac{1}{2}(\frac{1}{2}+(1+B)\eta)\log(-(1+B))+B\eta\log(-B).\end{aligned}$$

The proofs of Proposition 3.1 and Proposition 3.2 will also be given in [1] (see also [2], [3] for the idea of the proof.). There is an invertible matrix

$$\begin{pmatrix} d_{11} & d \\ d_{21} & 0\end{pmatrix} \tag{4.12}$$

which relates (w_0, w_1) to $(\Psi_{+}^{\mathrm{II}}, \Psi_{-}^{\mathrm{II}})$:

$$\begin{pmatrix} w_0, w_1\end{pmatrix} = \begin{pmatrix}\Psi_{+}^{\mathrm{II}}, \Psi_{-}^{\mathrm{II}}\end{pmatrix}\begin{pmatrix} d_{11} & d \\ d_{21} & 0\end{pmatrix}, \tag{4.13}$$

where

$$d=\sqrt{-\frac{A\eta}{2}}\mathrm{e}^{\frac{\pi i}{2}(1-A\eta)}\mathrm{e}^{-h_0}\mathrm{e}^{V_0^D}.$$

This tells us that what we have to do to prove Theorem 3.1 is to determine d_{11} and d_{21}. We consider an analytic continuation of the right-hand side of (4.13) from the region II to the region IV encircling $z=z_0$ in a clockwise manner. By using the connection formula for the Borel sums of WKB solutions (cf. [15], Theorem 2.23), the analytic continuation of the right-hand side becomes

$$\begin{pmatrix}\Psi_{+}^{\mathrm{IV}}, \Psi_{-}^{\mathrm{IV}}\end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 0\end{pmatrix}\begin{pmatrix} d_{11} & d \\ d_{21} & 0\end{pmatrix}.$$

Therefore we get

$$\begin{pmatrix} w_0, w_1\end{pmatrix} = \begin{pmatrix}\Psi_{+}^{\mathrm{IV}}, \Psi_{-}^{\mathrm{IV}}\end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 0\end{pmatrix}\begin{pmatrix} d_{11} & d \\ d_{21} & 0\end{pmatrix}. \tag{4.14}$$

On the other hand, u_j $(j=0,1)$ are related to $\Psi_{\pm}^{\mathrm{IV}}$ via the following invertible matrix:

$$\begin{pmatrix} d'_{11} & d' \\ d'_{21} & 0 \end{pmatrix} \tag{4.15}$$

as

$$\begin{pmatrix} u_0, u_1 \end{pmatrix} = \begin{pmatrix} \Psi_{+}^{\mathrm{IV}}, \Psi_{-}^{\mathrm{IV}} \end{pmatrix} \begin{pmatrix} d'_{11} & d' \\ d'_{21} & 0 \end{pmatrix}, \tag{4.16}$$

where

$$d' = \sqrt{-\frac{B\eta}{2}}\mathrm{e}^{-\frac{\pi i}{2}(1-B\eta)}\mathrm{e}^{-h_1}\mathrm{e}^{V_1^D}. \tag{4.17}$$

Here we use the relation between w_j and u_j $(j=0,1)$ (Cf. [11], [14]):

$$\begin{pmatrix} w_0, w_1 \end{pmatrix} = \begin{pmatrix} u_0, u_1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}, \tag{4.18}$$

where

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} \dfrac{\Gamma(1+A\eta)\Gamma(-B\eta)}{\Gamma(1+(A+1)\eta)\Gamma(-(B+1)\eta)} & \dfrac{\Gamma(1-A\eta)\Gamma(-B\eta)}{\Gamma(1+\eta)\Gamma(-(A+B+1)\eta)} \\ \dfrac{\Gamma(1+A\eta)\Gamma(B\eta)}{\Gamma(-\eta)\Gamma(1+(A+B+1)\eta)} & \dfrac{\Gamma(1-A\eta)\Gamma(B\eta)}{\Gamma(-(1+A)\eta)\Gamma(1+(1+B)\eta)} \end{pmatrix}.$$

Combining (4.14), (4.16) and (4.18), we obtain

$$\begin{pmatrix} 1 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} d_{11} & d \\ d_{21} & 0 \end{pmatrix} = \begin{pmatrix} d'_{11} & d' \\ d'_{21} & 0 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}. \tag{4.19}$$

By solving (4.19), we have

$$w_0 = \frac{u_{11}}{u_{12}}(d\Psi_{+}^{\mathrm{II}} + \frac{1}{i}d'(u_{22} - \frac{u_{12}u_{21}}{u_{11}})\Psi_{-}^{\mathrm{II}}). \tag{4.20}$$

Thus we have Theorem 3.1. □

Remark. In the case where $\eta = n$, the element u_{21} in the right-hand side of (4.19) will vanish. This imply that the equation (4.20) can be reduced to the following equation:

$$w_0 = \frac{u_{11}}{u_{12}}(d\Psi_{+}^{\mathrm{II}} + \frac{1}{i}d'u_{22}\Psi_{-}^{\mathrm{II}}) \tag{4.21}$$

Since Corollary 3.1 can be easily shown by using the relation (2.3) and Watson's lemma [25], in the rest of this paper, we show that (4.3) coincides with the corresponding equation in [17]. The right hand side of (4.3) can be rewritten as follows:

(4.22)
$$\begin{aligned}
&2^n z^{-\frac{1}{2}(1+nA)}(1-z)^{-\frac{1}{2}(1+nB)}\frac{\Gamma(1+n(1+A+B))}{\Gamma(1+n(2+A+B))}\\
&\times\frac{\Gamma(1+n)^{\frac{1}{2}}\Gamma(1-nB)\Gamma(nB)\Gamma(1+n(A+1))^{\frac{1}{2}}\Gamma(-n(1+A+B))^{\frac{1}{2}}}{\Gamma(1-n(2+A+B))\Gamma(n(2+A+B))\Gamma(-n(1+B))\Gamma(1+n(1+B))^{\frac{1}{2}}}\\
&\times\Big[e^{\lambda_1}\frac{\Gamma(1-n(2+A+B))\Gamma(n(2+A+B))}{\Gamma(nB)\Gamma(1-nB)}\psi_+\\
&\qquad\qquad+\frac{1}{i}e^{\lambda_2}\frac{\Gamma(1-n(2+A+B))\Gamma(n(2+A+B))}{\Gamma(1+n(1+A))\Gamma(-n(1+A))}\psi_-\Big].
\end{aligned}$$

By using Stirling's formula [21]:

$$\Gamma(z)\sim\sqrt{2\pi}e^{-z}z^{z-\frac{1}{2}}\ (|z|\to\infty,\ |\arg z|\le\pi-\delta)$$

and the well-known formula

$$\Gamma(1-z)\Gamma(z)=\pi/\sin(\pi z),$$

we have

$$\begin{aligned}
&2^{n-\frac{1}{2}}n^{\frac{1}{2}}e^{-\frac{n\pi i}{2}}z^{-\frac{1}{2}(1+nA)}(1-z)^{-\frac{1}{2}(1+nB)}\\
&\times\frac{(A+1)^{\frac{1}{2}(A+1)n+\frac{1}{4}}(B+1)^{\frac{1}{2}(B+1)n+\frac{1}{4}}(-A-B-1)^{\frac{1}{2}(A+B+1)n+\frac{1}{4}}}{(A+B+2)^{(A+B+2)n+\frac{1}{2}}}\\
&\times\Big[e^{\frac{\pi i}{4}-An\pi i}\frac{\sin(Bn\pi)}{\sin((A+B)n\pi)}\psi_++\frac{1}{i}e^{(\frac{3}{4}+2n)\pi i+Bn\pi i}\frac{\sin(An\pi)}{\sin((A+B)n\pi)}\psi_-\Big].
\end{aligned}\tag{4.23}$$

Since the WKB solution $\psi_\pm$ can be rewritten as

$$\begin{aligned}
\psi_+&=\frac{1}{\sqrt{S_{\rm odd}}}\exp\Big(\int_{z_0}^{z}S_{\rm odd}dz\Big)\\
&=\frac{1}{\sqrt{nS_{-1}}}\exp\Big(n\int_{z_0}^{z}S_{-1}dz\Big)\exp\Big(\int_{z_0}^{z}\frac{Q_1}{2S_{-1}}dz\Big)(1+\frac{1}{n}f(\frac{1}{n}))\\
&=e^{-\frac{\pi i}{4}}n^{-\frac{1}{2}}2^{-\frac{1}{2}}z^{\frac{1}{2}}(1-z)^{\frac{1}{2}}\frac{(A+B+2)^{\frac{1}{2}}}{((A+1)(B+1)(-A-B-1))^{\frac{1}{4}}}\\
&\times\frac{1}{i}\Bigg(\Big(\frac{z-z_1}{z-z_0}\Big)^{\frac{1}{4}}-\Big(\frac{z-z_0}{z-z_1}\Big)^{\frac{1}{4}}\Bigg)\exp\Big(n\int_{z_0}^{z}S_{-1}dz\Big)(1+\frac{1}{n}f(\frac{1}{n})),
\end{aligned}$$

$$
\begin{aligned}
\psi_- &= \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(-\int_{z_0}^{z} S_{\mathrm{odd}} dz\right) \\
&= \frac{1}{\sqrt{nS_{-1}}} \exp\left(-n\int_{z_0}^{z} S_{-1} dz\right) \exp\left(-\int_{z_0}^{z} \frac{Q_1}{2S_{-1}} dz\right)(1+\frac{1}{n}g(\frac{1}{n})) \\
&= \mathrm{e}^{-\frac{\pi i}{4}} n^{-\frac{1}{2}} 2^{-\frac{1}{2}} z^{\frac{1}{2}} (1-z)^{\frac{1}{2}} \frac{(A+B+2)^{\frac{1}{2}}}{((A+1)(B+1)(-A-B-1))^{\frac{1}{4}}} \\
&\times \left(\left(\frac{z-z_0}{z-z_1}\right)^{\frac{1}{4}} + \left(\frac{z-z_1}{z-z_0}\right)^{\frac{1}{4}}\right) \exp\left(-n\int_{z_0}^{z} S_{-1} dz\right)(1+\frac{1}{n}g(\frac{1}{n})),
\end{aligned}
$$

where $f(\frac{1}{n})$ and $g(\frac{1}{n})$ denote formal power series in $\frac{1}{n}$. Then (4.23) becomes

(4.24)

$$
\begin{aligned}
& 2^n z^{-\frac{An}{2}} (1-z)^{-\frac{Bn}{2}} \mathrm{e}^{-\frac{n\pi i}{2}} \\
& \times \frac{(A+1)^{\frac{1}{2}(A+1)n} (B+1)^{\frac{1}{2}(B+1)n} (-A-B-1)^{\frac{1}{2}(A+B+1)n}}{(A+B+2)^{(A+B+2)n}} \\
& \times \Bigg[\mathrm{e}^{-An\pi i} \frac{\sin(Bn\pi)}{\sin((A+B)n\pi)} \frac{1}{2i} \left(\left(\frac{z-z_1}{z-z_0}\right)^{\frac{1}{4}} - \left(\frac{z-z_0}{z-z_1}\right)^{\frac{1}{4}}\right) \mathrm{e}^{\left(n\int_{z_0}^{z} S_{-1} dz\right)} (1+\frac{1}{n}f(\frac{1}{n})) \\
& \quad + \mathrm{e}^{Bn\pi i} \frac{\sin(An\pi)}{\sin((A+B)n\pi)} \frac{1}{2} \left(\left(\frac{z-z_1}{z-z_0}\right)^{\frac{1}{4}} + \left(\frac{z-z_0}{z-z_1}\right)^{\frac{1}{4}}\right) \mathrm{e}^{\left(-n\int_{z_0}^{z} S_{-1} dz\right)} (1+\frac{1}{n}g(\frac{1}{n})) \Bigg].
\end{aligned}
$$

Here the constant e^{-nc} which appears in [17] has the following form:

(4.25)

$$
\mathrm{e}^{-nc} = 2^{\frac{(A+B+2)n}{2}} \mathrm{e}^{-\frac{A+1}{2} n\pi i} \frac{(A+1)^{\frac{1}{2}(A+1)n} (B+1)^{\frac{1}{2}(B+1)n} (-A-B-1)^{\frac{1}{2}(A+B+1)n}}{(A+B+2)^{(A+B+2)n}}.
$$

Using the above e^{-nc} to rewrite (4.24) and going back to x variable, we have

$$
\begin{aligned}
& \mathrm{e}^{-nc} (x-1)^{-\frac{An}{2}} (x+1)^{-\frac{Bn}{2}} \\
& \quad \times \Bigg[\mathrm{e}^{-An\pi i} \frac{\sin(Bn\pi)}{\sin((A+B)n\pi)} \frac{1}{2} \left(\left(\frac{x-\zeta_-}{x-\zeta_+}\right)^{\frac{1}{4}} + \left(\frac{x-\zeta_+}{x-\zeta_-}\right)^{\frac{1}{4}}\right) \mathrm{e}^{n\phi(x)} (1+\frac{1}{n}f(\frac{1}{n})) \\
& \qquad + \mathrm{e}^{Bn\pi i} \frac{\sin(An\pi)}{\sin((A+B)n\pi)} \frac{1}{2i} \left(\left(\frac{x-\zeta_-}{x-\zeta_+}\right)^{\frac{1}{4}} - \left(\frac{x-\zeta_+}{x-\zeta_-}\right)^{\frac{1}{4}}\right) \mathrm{e}^{-n\phi(x)} (1+\frac{1}{n}g(\frac{1}{n})) \Bigg].
\end{aligned}
$$

The expression (4.26) coincides with the right-hand side of the corresponding asymptotic expansion (c) in [17].

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